



# Photon propagator in light-shell gauge

## Citation

Georgi, Howard, Greg Kestin, and Aqil Sajjad. 2016. "Photon Propagator in Light-Shell Gauge." *Phys. Rev. D* 93 (9) (May 4). doi:10.1103/physrevd.93.096002.

## Published Version

doi:10.1103/PhysRevD.93.096002

## Permanent link

<http://nrs.harvard.edu/urn-3:HUL.InstRepos:28410203>

## Terms of Use

This article was downloaded from Harvard University's DASH repository, and is made available under the terms and conditions applicable to Open Access Policy Articles, as set forth at <http://nrs.harvard.edu/urn-3:HUL.InstRepos:dash.current.terms-of-use#OAP>

## Share Your Story

The Harvard community has made this article openly available.  
Please share how this access benefits you. [Submit a story](#).

[Accessibility](#)

# THE PHOTON PROPAGATOR IN LIGHT-SHELL GAUGE

Howard Georgi,<sup>1</sup> Greg Kestin,<sup>2</sup> Aqil Sajjad,<sup>3</sup>

Center for the Fundamental Laws of Nature  
Jefferson Physical Laboratory  
Harvard University  
Cambridge, MA 02138

## Abstract

We derive the photon propagator in light-shell gauge (LSG), introduced in [1] in the context of light-shell effective theory.

arXiv:1312.1741v1 [hep-ph] 6 Dec 2013

---

<sup>1</sup>georgi@physics.harvard.edu

<sup>2</sup>kestin@physics.harvard.edu

<sup>3</sup>sajjad@physics.harvard.edu

# 1 Introduction

In this paper we calculate the photon propagator in what we have called light-shell gauge (LSG). The motivation for working in LSG is described in depth in [1] (and [2]) where we discuss light-shell effective theory (LSET). We hope that LSET, for which LSG is an essential ingredient, may eventually provide another way of looking at high-energy scattering in gauge theories.

While perturbative computations in gauge theories are most commonly carried out in covariant gauges, where the procedure has been well established [3] there *are* venues in which non-covariant gauges may be preferable. In this paper we derive the photon propagator in light-shell gauge, which is defined by the condition

$$v_\mu A^\mu = 0 \quad (1)$$

where

$$v^\mu = (1, \hat{r})^\mu \quad (2)$$

So, in terms of the scalar potential  $A^0$  and the components of  $\vec{A}$ , (1) can be written as

$$A^0 = A_r \quad (3)$$

where

$$A_r \equiv \hat{r} \cdot \vec{A} \quad (4)$$

is the radial component of  $\vec{A}$  (and is not to be confused with  $A_r$  in the covariant tensor form). Note also that because (2) is not well-defined at the position space origin, many of our subsequent manipulations are ill-defined there, and we expect our propagator to make sense only in the punctured space from which the origin is excluded.

A gauge that shares some characteristics with LSG is radial (Fock-Schwinger) gauge [4] which is defined by the condition

$$x_\mu A^\mu = 0, \quad (5)$$

and has found widespread use in QCD sum-rules [5]. Shared characteristics between LSG and radial gauge include breaking translational invariance by choosing an origin and **coordinate dependent gauge condition**. As a result, it is often convenient to use a position space formulation rather than momentum space formulation. While these gauges share some characteristics, only LSG guarantees zero field strength off of the light-shell [1] and allows for simplification of calculations in LSET [2]. Another important difference is that the radial gauge condition is invariant under homogeneous Lorentz transformations, while LSG is only invariant under rotations about the origin.

Since we are at such an early stage (the first, as far as we know) in exploring this gauge, we restrict our analysis to QED where we can avoid complications that come with non-abelian theories.<sup>4</sup> Even in QED, we cannot use standard techniques for calculating propagators in non-covariant gauges, such as LSG. We therefore, along the road to the LSG propagator, present a different derivation which we hope may prove useful in other gauges as well.

The basic outline of our derivation is as follows. We begin by writing the photon lagrangian in LSG in a matrix form, treating  $\vec{A}$  and  $\hat{r}$  as column vectors. In particular, we show that the photon's kinetic energy can be written

$$\mathcal{L} = -\frac{1}{2} \begin{pmatrix} A_r & \vec{A}_\perp^T \end{pmatrix} M \begin{pmatrix} A_r \\ \vec{A}_\perp \end{pmatrix} \quad (6)$$

where we treat  $\vec{A}$  as a column vector and write

$$\vec{A}_\perp = \vec{A} - \hat{r} \hat{r}^T \vec{A} = \vec{A} - (\hat{r} \cdot \vec{A}) \hat{r} \quad (7)$$

Then in the following sections we will show how from  $M$  we are able to construct the LSG propagator. This is not simply a matter of inverting  $M$  because  $\vec{A}_\perp$  does not have a radial component. What we therefore need to compute is the inverse of  $M$  restricted to the subspace from which we have projected out this (non-existent) radial component. We will see that doing so turns out to be non-trivial since  $M$  does not commute with the projection operator in the radial direction. As a result, we cannot express  $M$  in a diagonal basis and simply take the inverse on the relevant subspace to obtain the propagator. We therefore need to follow a slightly more involved procedure. Our technique, we hope, may also be applicable to other non-covariant gauges.

---

<sup>4</sup>We hope to extend this work to QCD and in the process describe attributes avoided herein (e.g. ghosts).

## 2 The Lagrangian in LSG

We will now find the matrix  $M$  in equation (6) starting with the standard form of the photon kinetic energy:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\left(\vec{\nabla}A^0 + \partial_t\vec{A}\right)^2 - \frac{1}{2}\left(\vec{\nabla} \times \vec{A}\right)^2 \quad (8)$$

We then insert the LSG condition  $A^0 = A_r$ , giving

$$\mathcal{L} = \frac{1}{2}\left(\vec{\nabla}A_r + \partial_t\vec{A}\right)^2 - \frac{1}{2}\left(\vec{\nabla} \times \vec{A}\right)^2 \quad (9)$$

To arrive at the form given in (6), we manipulate the above terms one at a time. The first term can be written as

$$\left(\partial_t\vec{A} + \vec{\nabla}A_r\right)^2 = \left(\hat{r}\left(\partial_t + \hat{r}\cdot\vec{\nabla}\right)A_r + \vec{\nabla}_\perp A_r + \partial_t\vec{A}_\perp\right)^2 \quad (10)$$

where (not yet in a matrix notation)

$$\vec{\nabla}_\perp = \vec{\nabla} - \hat{r}\left(\hat{r}\cdot\vec{\nabla}\right) \quad (11)$$

We expand (10) to get

$$= \left(\left(\partial_t + \hat{r}\cdot\vec{\nabla}\right)A_r\right)^2 + \left(\partial_t\vec{A}_\perp\right)^2 + \left(\vec{\nabla}_\perp A_r\right)^2 + 2\left(\vec{\nabla}_\perp A_r\right)\cdot\partial_t\vec{A}_\perp \quad (12)$$

Integrating this by parts gives

$$\begin{aligned} &= -A_r(\partial_t + \vec{\nabla}\cdot\hat{r})(\partial_t + \hat{r}\cdot\vec{\nabla})A_r - A_r\vec{\nabla}^2 A_r + A_r\left(\vec{\nabla}\cdot\hat{r}\right)\left(\hat{r}\cdot\vec{\nabla}\right)A_r \\ &\quad - \vec{A}_\perp\cdot\partial_t^2\vec{A}_\perp - A_r\partial_t\vec{\nabla}_\perp\cdot A_\perp - \vec{A}_\perp\cdot\vec{\nabla}_\perp\partial_t A_r \end{aligned} \quad (13)$$

For the  $\left(\vec{\nabla} \times \vec{A}\right)^2$  term we can write

$$\left(\vec{\nabla} \times \vec{A}\right)^2 = \left(\vec{\nabla} \times A_r\hat{r} + \vec{\nabla} \times \vec{A}_\perp\right)^2 \quad (14)$$

We can work out the  $rr$ ,  $r \perp$ ,  $\perp r$  and  $\perp\perp$  terms in this separately by writing all the cross products explicitly in terms of Cartesian indices and simplifying. The  $rr$  term is

$$\left(\vec{\nabla} \times A_r\hat{r}\right)^2 = \left(\hat{r} \times \vec{\nabla}A_r\right)^2 = (\hat{r}_j\nabla_k A_r)(\hat{r}_j\nabla_k A_r) - (\hat{r}_j\nabla_k A_r)(\hat{r}_k\nabla_j A_r) \quad (15)$$

$$= \left(\vec{\nabla}A_r\right)^2 - (\hat{r}_k\nabla_k A_r)(\hat{r}_j\nabla_j A_r) \quad (16)$$

$$= \left(\vec{\nabla}A_r\right)^2 - \left(\hat{r}\cdot\vec{\nabla}A_r\right)^2 \quad (17)$$

Integrating this by parts gives

$$= -A_r\nabla^2 A_r + A_r\left(\vec{\nabla}\cdot\hat{r}\right)\left(\hat{r}\cdot\vec{\nabla}\right)A_r \quad (18)$$

The  $\perp\perp$  term is

$$\left(\vec{\nabla} \times \vec{A}_\perp\right)\cdot\left(\vec{\nabla} \times \vec{A}_\perp\right) = (\nabla_j A_\perp^k)(\nabla_j A_\perp^k) - (\nabla_j A_\perp^k)(\nabla_k A_\perp^j) \quad (19)$$

$$= -\vec{A}_\perp\cdot\nabla^2\vec{A}_\perp + \left(\vec{A}_\perp\cdot\vec{\nabla}\right)\left(\vec{\nabla}\cdot\vec{A}_\perp\right) \quad (20)$$

Similarly, it can be shown that the  $r \perp$  and  $\perp r$  terms are

$$\left(\vec{\nabla} \times A_r\hat{r}\right)\cdot\left(\vec{\nabla} \times \vec{A}_\perp\right) = A_r\left(\vec{\nabla}\cdot\hat{r}\right)\left(\vec{\nabla}\cdot\vec{A}_\perp\right) \quad (21)$$

and

$$\left(\vec{\nabla} \times \vec{A}_\perp\right)\cdot\left(\vec{\nabla} \times A_r\hat{r}\right) = \left(\vec{A}_\perp\cdot\vec{\nabla}\right)\left(\hat{r}\cdot\vec{\nabla}A_r\right) \quad (22)$$

Combining all the terms from (13), (18), (20), (21), and (22), we can write the Lagrangian in the matrix form in (6) repeated below

$$\mathcal{L} = -\frac{1}{2} \begin{pmatrix} A_r & \vec{A}_\perp^T \end{pmatrix} M \begin{pmatrix} A_r \\ \vec{A}_\perp \end{pmatrix} \quad (23)$$

where we now know the matrix  $M$  is given by

$$M = \begin{pmatrix} (\partial_t + \vec{\nabla} \cdot \hat{r}) \\ \vec{\nabla} \end{pmatrix} ((\partial_t + \hat{r} \cdot \vec{\nabla}) \quad \vec{\nabla}^T) + \begin{pmatrix} 0 & 0 \\ 0 & I \square \end{pmatrix} \quad (24)$$

Now things get a little complicated. The  $4 \times 4$  matrix differential operator  $M$  is invertible, but its inverse is not the propagator we want. The LSG propagator is the inverse of  $M$  restricted to the subspace from which we have projected out the (non-existent) radial component of  $\vec{A}_\perp$ . Let  $P$  be the projection operator onto the radial direction of  $\vec{A}$ . Then the inverse we are looking for is the operator  $D$  satisfying

$$\begin{aligned} P D &= D P = 0 \\ (I - P) M (I - P) D &= D (I - P) M (I - P) = (I - P) \end{aligned} \quad (25)$$

Because  $P$  does not commute with  $M$ , we cannot simply invert  $M$  and then project onto the relevant subspace. Instead, we will use a 2-step procedure. We will first show how the linear algebra of this 2-step procedure works in general, and then apply it to the LSG propagator in particular.

### 3 Inversion on a subspace

Our aim is to take an invertible matrix  $M$ , and find its inverse restricted to the subspace projected onto by  $(I - P)$ , where  $P$  is a projection operator onto a subspace and  $I$  is the identity matrix. That is, we wish to find the matrix  $D$  satisfying (25). There are two steps. Step one (which, for LSG, we will put off until later and relegate to an appendix) is to find the inverse of  $M^{-1}$  on the space projected onto by  $P$ . That is, we find an operator  $\nu$  satisfying

$$\nu P = P \nu = \nu \quad \nu P M^{-1} P = P M^{-1} P \nu = P \quad (26)$$

Then in step two we consider the following operator:

$$D = M^{-1} - M^{-1} \nu M^{-1} = M^{-1} - M^{-1} P \nu P M^{-1}, \quad (27)$$

It is straightforward to apply (26) to see that  $D$  satisfies (25), and thus it is the desired inversion of  $M$  on the subspace projected by  $(I - P)$ .

### 4 Returning to the LS gauge propagator

We now show how we can apply (27) to find the LSG propagator. In this and the following sections we will use an operator notation (discussed in more detail in appendix A) in which differential operators, their inverses, and ordinary functions of coordinates are all treated as linear operators acting on the tensor product space of our 4-component index space and the space of functions of the coordinates.

In this language, the projection operator  $P$  is

$$P = \begin{pmatrix} 0 & 0 \\ 0 & \hat{R} \hat{R}^T \end{pmatrix} \quad (28)$$

Since the formula (27) for the inverse on a subspace involves the inverse of  $M$  on the full space, we must begin by finding  $M^{-1}$ . For this purpose, it is convenient to note that  $M$  can be written in terms of a diagonal matrix  $M_d$  and a triangular matrix  $T$  as (where  $I_n$  is the  $n \times n$  identity operator)

$$M = T M_d T^\dagger, \quad (29)$$

where

$$M_d = \begin{pmatrix} \left( \partial_t + \vec{\nabla}^T \hat{R} \right) \left( \partial_t + \hat{R}^T \vec{\nabla} \right) & 0 \\ 0 & \square \end{pmatrix}, \quad (30)$$

$$T = \begin{pmatrix} 1 & 0 \\ \vec{\nabla} \left( \partial_t + \vec{\nabla}^T \hat{R} \right)^{-1} & I_3 \end{pmatrix} \quad (31)$$

and

$$T^\dagger = \begin{pmatrix} 1 & \left( \partial_t + \hat{R}^T \vec{\nabla} \right)^{-1} \vec{\nabla}^T \\ 0 & I_3 \end{pmatrix} \quad (32)$$

This makes inverting  $M$  straightforward, and we get for  $M^{-1}$

$$\begin{pmatrix} \left( \partial_t + \hat{R}^T \vec{\nabla} \right)^{-1} \left( 1 + \vec{\nabla}^T \square^{-1} \vec{\nabla} \right) \left( \partial_t + \vec{\nabla}^T \hat{R} \right)^{-1} & - \left( \partial_t + \hat{R}^T \vec{\nabla} \right)^{-1} \vec{\nabla}^T \square^{-1} \\ - \square^{-1} \vec{\nabla} \left( \partial_t + \vec{\nabla}^T \hat{R} \right)^{-1} & \square^{-1} \end{pmatrix} \quad (33)$$

The next ingredient we need is the inverse of  $M^{-1}$  restricted to the subspace. Here it is useful to avoid the matrix structure and define a linear operator  $\mu$ , as

$$\mu = \begin{pmatrix} 0 & \hat{R}^T \end{pmatrix} M^{-1} \begin{pmatrix} 0 \\ \hat{R} \end{pmatrix} \quad (34)$$

whence  $\nu$  in (26) is given by

$$\nu = \begin{pmatrix} 0 & 0 \\ 0 & \hat{R} \mu^{-1} \hat{R}^T \end{pmatrix} \quad (35)$$

Now we can just use (27) and put the pieces together to formally compute the LSG propagator. Doing so and simplifying gives the following results:

$$D_{rr} = \left( \partial_t + \hat{R}^T \vec{\nabla} \right)^{-1} \left( 1 + \vec{\nabla}^T C \vec{\nabla} \right) \left( \partial_t + \vec{\nabla}^T \hat{R} \right)^{-1} \quad (36)$$

$$D_{r\perp} = - \left( \partial_t + \hat{R}^T \vec{\nabla} \right)^{-1} \vec{\nabla}^T C \quad (37)$$

$$D_{\perp r} = -C \vec{\nabla} \left( \partial_t + \vec{\nabla}^T \hat{R} \right)^{-1} \quad (38)$$

$$D_{\perp\perp} = C \quad (39)$$

where  $C$  is given by

$$C = \square^{-1} - \square^{-1} \hat{R} \mu^{-1} \hat{R}^T \square^{-1} \quad (40)$$

Note that from this form we can see that  $C$  is transverse; that is, if we act with the projection operator for the transverse subspace on either side of  $C$ , we get  $C$ . What remains to be done is to derive an explicit form for  $C$ , which is done in detail in appendix B, with the result

$$C = -R \vec{\nabla}_\perp \square^{-1} L^{-2} R \vec{\nabla}_\perp^T + \vec{L} \square^{-1} L^{-2} \vec{L}^T \quad (41)$$

where

$$R \equiv |\vec{R}|. \quad (42)$$

Since this involves  $L^{-2}$ , we must show that this is well defined. We show in appendix C that because of the operators that appear on either side of  $L^{-2}$  in (41), the  $L^{-2}$  operator never acts on an  $L = 0$  state, and the expression (41) makes sense.

Putting (41) into (36-39) gives

$$D_{rr} = \left( \partial_t + \hat{R}^T \vec{\nabla} \right)^{-1} \left( 1 - R^{-1} L^2 \square^{-1} R^{-1} \right) \left( \partial_t + \vec{\nabla}^T \hat{R} \right)^{-1} \quad (43)$$

$$D_{r\perp} = \left( \partial_t + \hat{R}^T \vec{\nabla} \right)^{-1} R^{-1} \square^{-1} \vec{\nabla}_\perp^T R \quad (44)$$

$$D_{\perp r} = R \vec{\nabla}_\perp \square^{-1} R^{-1} \left( \partial_t + \vec{\nabla}^T \hat{R} \right)^{-1} \quad (45)$$

$$D_{\perp\perp} = -R \vec{\nabla}_\perp \square^{-1} L^{-2} R \vec{\nabla}_\perp^T + \vec{L} \square^{-1} L^{-2} \vec{L}^T \quad (46)$$

We can also combine these into a  $3 \times 3$  matrix form, call it  $D_3$ , appropriate for unconstrained  $\vec{A}$  fields:

$$\begin{aligned} & \hat{R} \left( \partial_t + \hat{R}^T \vec{\nabla} \right)^{-1} \left( \partial_t + \vec{\nabla}^T \hat{R} \right)^{-1} \hat{R}^T + \vec{L} \square^{-1} L^{-2} \vec{L}^T \\ & - \left( R \vec{\nabla}_\perp L^{-2} - \hat{R} \left( \partial_t + \hat{R}^T \vec{\nabla} \right)^{-1} R^{-1} \right) L^2 \square^{-1} \left( L^{-2} R \vec{\nabla}_\perp^T - R^{-1} \left( \partial_t + \vec{\nabla}^T \hat{R} \right)^{-1} \hat{R}^T \right) \end{aligned} \quad (47)$$

## 5 Conclusion

Here we have derived the photon propagator in light-shell gauge. In the process of this derivation, we have presented a technique that may also be useful for calculations in other non-covariant gauges (and, we hope, other applications). LSG is a crucial part of the construction of the light-shell effective theory [1], which we hope may provide a new viewpoint for high-energy scattering in gauge theories. We also hope that further insight can be gained once this method is extended to non-abelian gauge theories.

## Acknowledgements

We have benefited greatly from suggestions by Matthew Schwartz, Benjamin Grinstein, Randall Kelley, Aneesh Manohar and David Simmons-Duffin. Some of the initial work on this project by HG was done at the Aspen Center for Physics. He is grateful for the support of the Center and National Science Foundation grant #1066293. This research has been supported at Harvard in part by the National Science Foundation under grants PHY-0804450 and PHY-1067976.

## References

- [1] H. Georgi, G. Kestin, and A. Sajjad, “Color Fields on the Light-Shell,” [arXiv:1004.1404 \[hep-ph\]](#).
- [2] H. Georgi, G. Kestin, and A. Sajjad, “Introducing Light-Shell Effective Theory,” *in preparation* (2013) .
- [3] P. Pascual and R. Tarrach, “QCD: Renormalization for the Practitioner,” *Springer* **194** (1984) .
- [4] S. Leupold, “Feynman rules in radial gauge,” [arXiv:hep-th/9609222](#).
- [5] M. A. Shifman, “Wilson loop in vacuum fields,” *Nucl. Phys.* **B173** (1980) .

## A Operator Notation

Throughout we have used a notation that involves local and non-local operators. For example, when a local operator, such as  $R^{-1}$  appears, it is

$$R^{-1}(x_1, x_2) = \frac{1}{r_1} \delta(x_1 - x_2) \quad (48)$$

and when not written, the delta function and integrations over the arguments are implicit. We also come across the operators  $\square^{-1}$ ,  $\left(\partial_t + \hat{R} \cdot \vec{\nabla}\right)^{-1}$  and  $\left(\partial_t + \vec{\nabla} \cdot \hat{R}\right)^{-1}$ . We know that  $\square^{-1}$  is the position space propagator for a massless scalar and is given by

$$\square^{-1}(x-y) = -\frac{i}{4\pi^2} \frac{1}{(x-y)^2 - i\epsilon} \quad (49)$$

To find the expression for  $(\partial_t + \hat{R} \cdot \nabla)^{-1}$ , we can consider the theory with the Lagrangian

$$\mathcal{L} = i\phi^*(\partial_t + \hat{r} \cdot \nabla)\phi \quad (50)$$

Using canonical quantization to find the 2-point function for this theory gives

$$(\partial_t + \hat{R} \cdot \vec{\nabla})^{-1} = \frac{1}{r'^2} \theta(t-t') \delta(t-r-t'+r') \delta(z-z') \delta(\phi-\phi') \quad (51)$$

Similarly

$$(\partial_t + \vec{\nabla} \cdot \hat{R})^{-1} = \frac{1}{r^2} \theta(t-t') \delta(t-r-t'+r') \delta(z-z') \delta(\phi-\phi') \quad (52)$$

## B Derivation of C

We can find  $C$  by brute force, but here we will instead use a slicker approach, which will take advantage of (25). Using the formula (24) for  $M$  and our result for the propagator (36)-(39), it is straightforward to see that  $(I_4 - P) M (I_4 - P) D$  is

$$\begin{pmatrix} 1 & 0 \\ \vec{\nabla}_\perp \left(\partial_t + \vec{\nabla}^T \hat{R}\right)^{-1} - (I_3 - P_3) \square C \vec{\nabla} \left(\partial_t + \vec{\nabla}^T \hat{R}\right)^{-1} & (I_3 - P_3) \square C \end{pmatrix} \quad (53)$$

where  $P_3 = \hat{R} \hat{R}^T$ , and we have used  $(I_3 - P_3) C = C$ . For  $D$  to be the LSG propagator, we want the 2nd row entries of  $(I - P) M (I - P) D$  to be 0 and  $I_3 - P_3$ . Both these requirements are satisfied if

$$(I_3 - P_3) \square C = I_3 - P_3 \quad (54)$$

We will now use this condition to find an explicit expression for  $C$ . Our approach will involve first finding a basis for the space perpendicular to  $\hat{R}$ , and then acting on (54) with various operators to find the components of  $C$  in this basis. We begin by identifying the proper basis. Notice that

$$R \vec{\nabla}_\perp^T = i \left( \vec{L} \times \hat{R} \right)^T \quad (55)$$

So  $\vec{L}$  and  $R \vec{\nabla}_\perp$  are both orthogonal to  $\hat{R}$  and orthogonal to one another, therefore forming our basis. We can express  $(I_3 - P_3)$  in terms of them. First note that from (55) it follows that

$$R \vec{\nabla}_\perp^T \vec{\nabla}_\perp R = -L^2 \quad (56)$$

so with proper normalization we have

$$(I_3 - P_3) = \vec{L} L^{-2} \vec{L}^T - \vec{\nabla}_\perp R L^{-2} R \vec{\nabla}_\perp^T \quad (57)$$

Now we want to find the components of  $C$ . The first, and easiest component to find is computed by acting on (54) with  $\vec{L}$  on both sides to give

$$\vec{L}^T (I_3 - P_3) \square C \vec{L} = \vec{L}^T (I_3 - P_3) \vec{L} \quad (58)$$

This is easy because  $\vec{L}$  commutes with  $\square$ , so we get

$$\vec{L}^T C \vec{L} = \square^{-1} L^2 \quad (59)$$

Acting on the left of (54) with  $\vec{L}^T$  and on the right with  $\vec{\nabla}_\perp R$  as follows

$$\vec{L}^T (I_3 - P_3) \square C \vec{\nabla}_\perp R = \vec{L}^T (I_3 - P_3) \vec{\nabla}_\perp R \quad (60)$$



works similarly once we observe  $\vec{L}^T \vec{\nabla}_\perp R = 0$ , giving

$$\vec{L}^T C \vec{\nabla}_\perp R = 0 \quad (61)$$

The final two matrix elements require the commutator

$$\left[ R \vec{\nabla}_\perp^T, \square \right] = 2R^{-2} L^2 \hat{R}^T \quad (62)$$

We now take a detour to demonstrate this commutator relation. We can write

$$\square = \partial_t^2 - \left( \vec{\nabla}^T \hat{R} \right) \left( \hat{R}^T \vec{\nabla} \right) + L^2 R^{-2} \quad (63)$$

The middle term in (63) can be written

$$\begin{aligned} \left( \vec{\nabla}^T \hat{R} \right) \left( \hat{R}^T \vec{\nabla} \right) &= \left( \vec{\nabla}^T \vec{R} \right) R^{-2} \left( \vec{R}^T \vec{\nabla} \right) \\ &= \left( \vec{R}^T \vec{\nabla} \right) R^{-2} \left( \vec{R}^T \vec{\nabla} \right) + 3R^{-2} \left( \vec{R}^T \vec{\nabla} \right) = R^{-2} \left( \left( \vec{R}^T \vec{\nabla} \right)^2 + \left( \vec{R}^T \vec{\nabla} \right) \right) \end{aligned} \quad (64)$$

We chose this particular form because  $\left( \vec{R}^T \vec{\nabla} \right)$  is a scaling operator that counts the total powers  $R$  or  $1/\vec{\nabla}$ .<sup>5</sup> So this term commutes with  $R \vec{\nabla}_\perp^T$  and the only term in  $\square$  that fails to commute is  $L^2 R^{-2}$ .

The factors of  $R$  commute with both  $\vec{\nabla}_\perp^T$  and  $L^2$ , so we just need to consider

$$\left[ R \vec{\nabla}_\perp^T, L^2 \right] \quad (65)$$

Using (55), we can write this in components, as

$$\left[ i \epsilon_{abc} L_b \hat{R}_c, L_d L_d \right] \quad (66)$$

$$= i \epsilon_{abc} L_b \left( L_d \left[ \hat{R}_c, L_d \right] + \left[ \hat{R}_c, L_d \right] L_d \right) \quad (67)$$

$$= -\epsilon_{abc} \epsilon_{cde} L_b \left( L_d \hat{R}_e + \hat{R}_e L_d \right) \quad (68)$$

$$= -L_b \left( \left[ L_a, \hat{R}_b \right] + 2\hat{R}_b L_a - 2L_b \hat{R}_a - \left[ \hat{R}_a, L_b \right] \right) \quad (69)$$

The first and fourth terms in (69) cancel each another. The second term vanishes because  $\vec{L} \cdot \hat{R} = 0$ . The third term gives

$$\left[ R \vec{\nabla}_\perp^T, L^2 \right] = 2L^2 \hat{R}^T \quad (70)$$

or

$$\left[ R \vec{\nabla}_\perp^T, \square \right] = 2R^{-2} L^2 \hat{R}^T \quad (71)$$

which is (62).

We now return to the derivation of  $C$ , but note that (71) vanishes when acting on  $C$ . So, acting with  $R \vec{\nabla}_\perp^T$  on the left and  $\vec{\nabla}_\perp R$  on the right gives

$$R \vec{\nabla}_\perp^T \square C \vec{\nabla}_\perp R = \square R \vec{\nabla}_\perp^T C \vec{\nabla}_\perp R = -L^2 \quad (72)$$

implying

$$R \vec{\nabla}_\perp^T C \vec{\nabla}_\perp R = -\square^{-1} L^2 \quad (73)$$

In the same way we can see that the last component is zero

$$R \vec{\nabla}_\perp^T \square C \vec{L} = \square R \vec{\nabla}_\perp^T C \vec{L} = 0 \quad (74)$$

Combining (59), (61), (73) and (74) with (57) gives

$$C = -R \vec{\nabla}_\perp \square^{-1} L^{-2} R \vec{\nabla}_\perp^T + \vec{L} \square^{-1} L^{-2} \vec{L}^T \quad (75)$$

---

<sup>5</sup>Note also that the last form is trivial to remember because it vanishes for  $r^a$  with  $a = 0$  or  $-1$  as it should.

## C Does $L^{-2}$ make sense?

The derivation of  $C$  (in appendix B) formally involves the inverse of  $L^2$ , and of course this makes no sense on  $L = 0$  states. But all we actually need is for (57) to make sense acting on arbitrary functions, so that

$$(I_3 - P_3) \vec{f}(\vec{r}) = \vec{L} L^{-2} \vec{L}^T \vec{f}(\vec{r}) - \vec{\nabla} R L^{-2} R \vec{\nabla}_\perp^T \vec{f}(\vec{r}) \quad (76)$$

This is perfectly well-defined, because if either the  $\vec{L}^T \vec{f}(\vec{r})$  or  $R \vec{\nabla}_\perp^T \vec{f}(\vec{r})$  component has zero angular momentum, then that component itself is zero. This can be seen by first noting that if  $L^2$  acting on either of these components is zero, then the component must be a function of the radius only, call it  $g(r)$ . If we integrate  $g(r)$  over  $d\Omega$ , we get  $4\pi g(r)$ , but at the same time we see that integrating either component over  $d\Omega$  must be zero because in both cases we are integrating a total derivative over a closed surface. Therefore  $g(r)$ , which denotes either component, is necessarily zero.